PMM U.S.S.R.,Vol.46,pp. 321-326

0021-8928/83/3 0321 $7.50/0

UDC 532.65

THE BRANCHING OF EQUILIBRIUM STATES OF AN ISOLATED ROTATING LIQUID MASS*

L.A. SLOBOZHANIN

Branching of solutions of the nonlinear problem of equilibrium of a rotating free mass of liquid possessing surface tension is investigated (expediency of such investigation was pointed out in /1,2/). It is shown that a two-parameter set of nonaxisymmetric states passes through the critical (in the sense of stability) axisymmetric equilibrium state. Form of the nonaxisymmetric equilibrium figures is analyzed, and properties of branching determined. The latter are used for assessing the stability of these figures.

1. Let us consider the simply connected forms of the free surface of a rotating isolated mass of viscous liquid held together by surface tension forces in the absence of an external force field. In cylindrical coordinates r, φ, z the form $r(s), z(s)$ of the axisymmetric equilibrium surface Γ is defined /3-5/ by the following differential equations, initial conditions, and the state of the given volume of liquid:

$$r'' = -z'\beta', \quad z'' = r'\beta', \quad \beta' = -pr^2 + q - z'/r \quad (p = \rho\omega^2/(2\sigma))$$
$$r(0) = 0, \quad r'(0) = 1, \quad z(0) = z_0, \quad z'(0) = 0$$
$$2\pi\int_0^\tau r^2(s; q)\, z_s'(s; q)\, ds = v$$

where s is the length of arc measured from the upper pole along cross section $\varphi = \text{const}$ of surface Γ, $\beta(s)$ is the angle of inclination to the horizontal of the tangent at points of that cross section, ω is the angular velocity of uniform rotation, ρ, σ are the density and coefficient of surface tension of the liquid, z_0 is the displacement (unimportant for the form) along the axis of rotation z, chosen on the basis of the condition of symmetry of the figure relative to plane $z = 0$ (Fig.1), and the before-hand unknown quantity q (the doubled mean curvature at the pole) is determined by the liquid mass volume v, and τ is the value of s at the equatorial point.

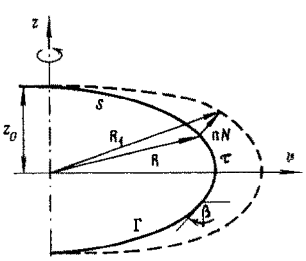

Fig.1

The axisymmetric figure form is uniquely determined by the quantity $\alpha = qp^{-1/3}$ and, as shown by calculations, if Γ is to be simply connected, it is necessary that $\alpha \leqslant \alpha_+ = 1.263$. As α increases, parameter $\nu = vp$ first increases from zero (when $\alpha = -\infty$) to $\nu_0 = 9.543$ (for $\alpha = 0.70$), and then decreases to $\nu_+ = 8.378$ (when $\alpha = \alpha_+$). In the section of increasing ν, $\alpha = 0$ corresponds to ν_+. Thus to each $\nu < \nu_+$ corresponds one axisymmetric equilibrium figure $(\alpha < 0)$, while two figures correspond to $\nu_+ \leqslant \nu < \nu_0$ which for $\nu = \nu_0$ merge into one $(\alpha = 0.70)$. When $\nu > \nu_0$ there are no simply connected axisymmetric figures.

On the other hand it is physically more natural to take as the parameter defining equilibrium not ν, but the quantity $\lambda = \mu^2/(\rho\sigma v^{7/3})$, where $\mu = I\omega$ is the moment of momentum of the liquid about the axis of rotation and I is the respective moment of inertia. It was shown in /2/ that as α increases from $-\infty$ to α_+, parameter λ monotonically increases from zero to 5.672.

2. To define the nonaxisymmetric simply connected equilibrium surface we shall first establish the correspondence between its points and the points on the axisymmetric surface Γ along the normal to the latter (Fig.1), i.e.

$$\mathbf{R}_1(s, \varphi) = \mathbf{R}(s, \varphi) + \mathbf{n}(s, \varphi)N(s, \varphi) \tag{2.1}$$

*Prikl.Matem.Mekhan.,46,No.3,pp.413-419,1982

$$\mathbf{R}(s,\ \varphi) = \mathbf{i}r(s)\cos\varphi + \mathbf{j}r(s)\sin\varphi + \mathbf{k}z(s)$$
$$\mathbf{n}(s,\ \varphi) = -\mathbf{i}z'(s)\cos\varphi - \mathbf{j}z'(s)\sin\varphi + \mathbf{k}r'(s)$$

The differential equation of such surface is of the form

$$2H(r,\ r',\ z',\ \beta',\ N,\ N_s',\ N_\varphi',\ N_{ss}'',\ N_{\varphi\varphi}'',\ N_{s\varphi}'') = -pr_1^2 + q, \qquad r_1 = r - z'N \tag{2.2}$$

The doubled curvature $2H$ is determined using the known formulas /6/ with (2.1) taken into account.

Function $N(s,\varphi)$ must satisfy besides EQ.(2.2) the condition of conservation of the volume of liquid and that of immobility of the center of mass

$$\int_\Gamma \left[N - \frac{1}{2}\left(\frac{z'}{r} + \beta'\right)N^2 + \frac{1}{3}\frac{z'}{r}\beta' N^3\right] d\Gamma = 0 \tag{2.3}$$
$$\int_\Gamma \left\{zN + \frac{1}{2}\left[r' - z\left(\frac{z'}{r} + \beta'\right)\right]N^2 + \frac{1}{3}\left[\frac{zz'}{r}\beta' - r'\left(\frac{z'}{r} + \beta'\right)\right]N^3 + \frac{1}{4}\frac{r'z'}{r}\beta' N^4\right\} d\Gamma = 0$$
$$\int_\Gamma \left[rN - \frac{1}{2}(2z' + r\beta')N^2 + \frac{1}{3}z'\left(\frac{z'}{r} + 2\beta'\right)N^3 - \frac{1}{4}\frac{z'^2}{r}\beta' N^4\right] \left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}\varphi\, d\Gamma = 0$$

and, also, to be bounded and 2π-periodic in φ.

3. The axisymmetric figures are stable for $p < p_* = 5.253v^{-1}$ /4,5/ (the shown value of p_* is that corrected in the course of calculations carried out below). The form of the critical axisymmetric surface is determined by $q = q_0 = -0.907p_*^{1/3}$. Let us investigate the problem of form of the equilibrium surfaces for values of parameter p close to critical. We select as the standard of comparison of Γ the axisymmetric critical surface and, in conformity with /5/ set

$$p = p_* \mp \varepsilon^2 \quad (\varepsilon > 0)$$
$$N = \varepsilon N_1(s,\ \varphi) + \varepsilon^2 N_2(s,\ \varphi) + \ldots,\quad q = q_0 + \varepsilon q_1 + \varepsilon^2 q_2 + \ldots \tag{3.1}$$

(the validity of these expansions can be strictly proved in the investigation of the problem by the Liapunov–Schmidt method /5,7/).

Substituting (3.1) into (2.2) and (2.3) and equating coefficients at like powers of ε, we obtain a sequence of problems in $N_i(s,\ \varphi)$. The problems for N_1 and N_2, and the equation for N_3 are

$$LN_1 = q_1,\quad \int N_1\, d\Gamma = 0,\quad \int N_1 z\, d\Gamma = \int N_1 r \left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}\varphi\, d\Gamma = 0 \tag{3.2}$$

$$LN_2 = -\left(\frac{z'^3}{r^3} + \beta'^3 + p_* z'^2\right)N_1^2 - 2r'\left(\frac{z'}{r^2} - p_* r\right)N_1 N_{1s}' + \frac{1}{2}\left(\frac{z'}{r} - \beta'\right)N_{1s}'^2 - 2\beta' N_1 N_{1ss}'' - \frac{1}{2r^2}\left(\frac{z'}{r} - \beta'\right)N_{1\varphi}'^2 - 2\frac{z'}{r'}N_1 N_{1\varphi\varphi}'' \mp r^2 + q_2 \tag{3.3}$$

$$\int N_2\, d\Gamma = \frac{1}{2}\int\left(\frac{z'}{r} + \beta'\right)N_1^2\, d\Gamma,\quad \int N_2 z\, d\Gamma = \frac{1}{2}\int N_1^2\left[z\left(\frac{z'}{r} + \beta'\right) - r'\right] d\Gamma \tag{3.4}$$
$$\int N_2 r\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}\varphi\, d\Gamma = \frac{1}{2}\int N_1^2(2z' + r\beta')\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right\}\varphi\, d\Gamma$$

$$\begin{aligned} LN_3 = {} & -2\left(\frac{z'^3}{r^3} + \beta'^3 + p_* z'^2\right)N_1 N_2 - 2r'\left(\frac{z'}{r^2} - p_* r\right)(N_1 N_{2s}' + N_2 N_{1s}') + \left(\frac{z'}{r} - \beta'\right)N_{1s}' N_{2s}' - \\ & 2\beta'(N_1 N_{2ss}'' + N_2 N_{1ss}'') - \frac{1}{r^2}\left(\frac{z'}{r} - \beta'\right)N_{1\varphi}' N_{2\varphi}' - \\ & 2\frac{z'}{r^3}(N_1 N_{2\varphi\varphi}'' + N_2 N_{1\varphi\varphi}'') - \left(\frac{z'^4}{r^4} + \beta'^4\right)N_1^3 - \\ & \left[\frac{r'}{r}\left(\frac{z'^2}{r^2} + \frac{z'}{r}\beta' + \beta'^2\right) + 3\beta'\beta''\right]N_1^2 N_{1s}' + \\ & \frac{1}{2}\left(\frac{z'^2}{r^2} + 2\frac{z'}{r}\beta' - 3\beta'^2\right)N_1 N_{1s}'^2 + \frac{1}{2}\frac{r'}{r}N_{1s}'^3 - \\ & \frac{1}{2r^2}\left(\frac{3z'^2}{r^2} - \frac{2z'}{r}\beta' - \beta'^2\right)N_1 N_{1\varphi}'^2 - \frac{1}{2}\frac{r'}{r^3}N_{1s}' N_{1\varphi}'^2 - \\ & 3\beta'^2 N_1^2 N_{1ss}'' + \frac{3}{2}N_{1s}'^2 N_{1ss}'' + \frac{1}{2r^2}N_{1\varphi}'^2 N_{1ss}'' - 3\frac{z'^2}{r^4}N_1^2 N_{1\varphi\varphi}'' + \\ & \frac{1}{2r^2}N_{1s}'^2 N_{1\varphi\varphi}'' + \frac{3}{2r^4}N_{1\varphi}'^2 N_{1\varphi\varphi}'' + \frac{2}{r^2}N_{1s}' N_{1\varphi}' N_{1s\varphi}'' \pm 2rz' N_1 + q_3 \end{aligned} \tag{3.5}$$

$$LN = N_{ss}'' + \frac{r'}{r}N_s' + \frac{1}{r^2}N_{\varphi\varphi}'' - aN,\quad a(s) = 2p_* rz' - \frac{z'^2}{r^2} - \beta'^2$$

where the integrals are taken over surface Γ.

4. Problem (3.2) is the same as the problem of the form of normal component of dangerous perturbations on the critical surface of Γ. Identifying all solutions obtained from one another by turning about the z axis, we obtain /4,5/

$$N_1 = Q_1 u(s) \sin 2\varphi, \quad q_1 = 0 \quad \left(u'' + \frac{r'}{r} u' - \left(a + \frac{4}{r^2}\right) u = 0\right) \tag{4.1}$$

where Q_1 is the unknown constant that is to be determined, $u(s)$ is the solution of the equation appearing in parentheses and bounded at the singular points $r = 0$ (when $s = 0$ and $s = 2\tau$). It is symmetric relative to the straight line $s = \tau\ (u(s) = u(2\tau - s))$, and in the zero neighborhood can be represented by the series

$$u(s) = \left[\frac{1}{2} s^2 + \left(\frac{1}{40} p_* q_0 + \frac{7}{23040} q_0^4\right) s^6 + \ldots\right] p_*^{-1/6}$$

We pass to the problem for N_2. Taking into account (4.1), we represent Eq.(3.3) in the form

$$LN_2 = f_2, \quad f_2 = Q_1^2 F_0(s) + Q_1^2 F_4(s) \cos 4\varphi \mp r^2 + q_2$$

$$F_0(s) = \left(\frac{3z'}{r^3} + \frac{\beta'}{r^2}\right) u^2 - F(s), \quad F_4(s) = \left(-\frac{5z'}{r^3} + \frac{\beta'}{r^2}\right) u^2 + F(s)$$

$$F(s) = \frac{1}{2}\left(\frac{z'^3}{r^3} + \beta'^3 + p_* z'^2\right) u^2 + r'\left(\frac{z'}{r^2} - p_* r\right) uu' - \frac{1}{4}\left(\frac{z'}{r} - \beta'\right) u'^2 + \beta' uu''$$

It can be readily checked that the orthogonality condition

$$\int_\Gamma f_2 N_1 \, d\Gamma = 0$$

is necessary and sufficient for the solvability of that equation.

For deriving the solution we introduce the operators

$$P_0 N_2 = \frac{1}{2\pi} \int_0^{2\pi} N_2(s, \varphi) \, d\varphi$$

$$P_k N_2 = \frac{1}{\pi} \left[\sin k\varphi \int_0^{2\pi} N_2(s, \varphi) \sin k\varphi \, d\varphi + \cos k\varphi \int_0^{2\pi} N_2(s, \varphi) \cos k\varphi \, d\varphi\right] \quad (k \geqslant 1)$$

and obtain

$$P_n L = L_n P_n, \quad L_n = \frac{\partial^2}{\partial s^2} + \frac{r'}{r} \frac{\partial}{\partial s} - a - \frac{n^2}{r^2} \quad (n \geqslant 0)$$

Hence by acting with operators P_n on (3.3) we obtain the equations

$$L_n P_n N_2 = P_n f_2$$

Representing N_2 in the form of the Fourier series

$$N_2 = \sum_{n=0}^{\infty} P_n N_2$$

and taking into account the form of function $f_2(s, \varphi)$, we obtain

$$N_2 = Q_1^2 g_1(s) \mp g_2(s) + q_2 g_3(s) + Q_1^2 g_4(s) \cos 4\varphi + Q_2 u(s) \sin 2\varphi \tag{4.2}$$

where Q_2 is an arbitrary constant and $g_j(s)\ (j = 1, 2, 3, 4)$ are solutions bounded at singular points $r = 0$ of equations

$$L_0 g_1 = F_0, \quad L_0 g_2 = r^2, \quad L_0 g_3 = 1, \quad L_4 g_4 = F_4 \tag{4.3}$$

which we represent in the form

$$g_i(s) = C_i u_0(s) + b_i(s) \quad (i = 1, 2, 3), \; g_4(s) = C_4 u_4(s) + b_4(s) \tag{4.4}$$
$$C_i = -b_i'(\tau)/u_0'(\tau), \quad C_4 = -b_4'(\tau)/u_4'(\tau)$$

where $b_j(s)\ (j = 1, 2, 3, 4)$, $u_0(s)$, $u_4(s)$ are the respective solutions of Eqs.(4.3), bounded when $s = 0$, and $L_0 u_0 = 0$, $L_4 u_4 = 0$. In the neighborhood of $s = 0$ they are of the form

$$b_1(s) = u_0(s) p_*^{-4/3}, \quad b_2(s) = u_0(s) p_*^{-4/3} + \frac{1}{16} s^4$$

$$b_3(s) = u_0(s) p_*^{-2/3} + \frac{1}{4}s^2 - \frac{1}{192}q_0^2 s^4, \qquad b_4(s) = u_4(s) p_*^{-4/3}$$
$$u_0(s) = 1 - \frac{1}{8}q_0^2 s^2 + \left(\frac{1}{8}p_* q_0 + \frac{1}{384}q_0^4\right)s^4, \qquad u_4(s) = \frac{1}{24}s^4$$

and are accurate within terms of order s^6.

Note that $u_0'(\tau)$ and $u_4'(\tau)$ are nonzero, Since otherwise the critical from the stability point of view would have been perturbations in fourth harmonics or axisymmetric perturbations, and not in second harmonics, as in fact is the case.

The symmetry of Γ relative to the equatorial plane, and the form of operators L_0, L_4 and functions $F_0(s)$, $F_4(s)$ imply that for the derived equations $g_j(s)$ the equality $g_j(s) = g_j(2\tau - s)$ holds. Hence these solutions are bounded not only when $s = 0$ but, also, for $s = 2\tau$.

The first of conditions (3.4) results in the relation (the remaining conditions of (3.4) are automatically satisfied)

$$q_2 = Q_1^2 D_1 \pm D_2, \qquad D_1 = \frac{1}{G_3}\left[\frac{1}{4}\int_0^\tau u^2(z' + r\beta')\,ds - G_1\right], D_2 = \frac{G_2}{G_3} \tag{4.5}$$
$$G_i = \int_0^\tau g_i r\,ds \qquad (i = 1, 2, 3)$$

Let us pass to Eq. (3.5) which we now write in the form $LN_3 = f_3$. The condition of its solvability

$$\int_\Gamma f_3 N_1\,d\Gamma = 0$$

with (4.1), (4.2), (4.4), and (4.5) yields the equality

$$Q_1^2(Q_1^2 B \pm A) = 0 \tag{4.6}$$
$$A = \int_0^\tau \{C_2(u_0\gamma_1 + u_0'\gamma_2 + 2u_0''\gamma_3) + b_2\gamma_1 + b_2'\gamma_2 + 2b_2''\gamma_3 +$$
$$2r^2 z' u^2 - D_2 E\}\,ds$$
$$B = \int_0^\tau \Big\{-C_1(u_0\gamma_1 + u_0'\gamma_2 + 2u_0''\gamma_3) +$$
$$\frac{1}{2}C_4(u_4\gamma_1 + u_4'\gamma_2 + 2u_4''\gamma_3 - 8u_4\gamma_4) - \left(b_1 - \frac{1}{2}b_4\right)\gamma_1 -$$
$$\left(b_1' - \frac{1}{2}b_4'\right)\gamma_2 - 2\left(b_1'' - \frac{1}{2}b_4''\right)\gamma_3 - 4b_4\gamma_4 - \left(\frac{3}{4}\frac{z'^4}{r^3} -\right.$$
$$\left.\frac{15}{2}\frac{z'^2}{r^3} + \frac{6}{r^3} - \frac{z'}{r^2}\beta' - \frac{1}{2r}\beta'^2 + \frac{3}{4}r\beta'^4\right)u^4 - \left(\frac{3}{4}\frac{r'z'^2}{r^2} +\right.$$
$$\left.3\frac{r'z'}{r}\beta' - \frac{3}{2}r'\beta'^2 + \frac{1}{2}\frac{r'}{r^2} - \frac{9}{2}p_* r^2 r'\beta'\right)u^3 u' +$$
$$\left[\frac{3}{8}r\left(\frac{z'^2}{r^3} + 2\frac{z'}{r}\beta' - 3\beta'^2\right) + \frac{1}{2r}\right]u^2 u'^2 + \frac{3}{8}r' u u'^3 +$$
$$\left(\frac{1}{2r} - \frac{9}{4}r\beta'^2\right)u^3 u'' + \frac{9}{8}r u u'^2 u'' - D_1 E\Big\}\,ds$$
$$E = C_3(u_0\gamma_1 + u_0'\gamma_2 + 2u_0''\gamma_3) + b_3\gamma_1 + b_3'\gamma_2 + 2b_3''\gamma_3$$
$$\gamma_1(s) = 2r\left[\left(\frac{z'^3}{r^3} - 4\frac{z'}{r^3} + \beta'^3 + p_* z'^2\right)u^2 +\right.$$
$$\left. r'\left(\frac{z'}{r^2} - p_* r\right)uu' + \beta' uu''\right]$$
$$\gamma_2(s) = r\left[2r'\left(\frac{z'}{r^2} - p_* r\right)u^2 + \left(\beta' - \frac{z'}{r}\right)uu'\right], \qquad \gamma_3(s) = r\beta' u^2$$
$$\gamma_4(s) = \frac{1}{r}\left(3\frac{z'}{r} + \beta'\right)u^2$$

Neglecting the solution $Q_1 = 0$, which relates to the axisymmetric form of equilibrium when p is close to p_*, from Eq. (4.6) we obtain

$$(Q_1)_1 = -(Q_1)_2 = (\mp A/B)^{1/2} \tag{4.7}$$

For calculating A and B we shall use everywhere dimensionless variables selecting the quantity $p_*^{-1/3}$ as the characteristic dimension of length. We retain the old notation for the new variables, since all previous relations remain valid by setting in them $p_* = 1$. The equations that determine the meridian cross section form Γ ($q = q_0 = -0.907$) was numerically calculated for s in steps of 10^{-4}, together with the determination of functions $u(s)$, $u_0(s)$, $u_4(s)$, $b_j(s)$ ($j = 1, 2, 3, 4$), followed by the determination of A and B. As the result we obtained $A = -1.660$, $B = -1.853$.

It follows now from (4.7) that the lower sign is to be taken at the A/B ratio and other similar places. Hence in conformity with the first of formulas (3.1) no branched-off nonaxisymmetric figures can exist when $p < p_*$.

Recalling the possibility of turning about the z axis, we conclude that a two-parameter set of nonaxisymmetric states positioned at $p < p_*$ and invariant to turns about the z axis pass through the critical axisymmetric equilibrium state. Deviation of the free surface form of the nonaxisymmetric from the critical surface form on the normal to the latter is determined in the first approximation by the dimensional quantity

$$N = 0.946\,(p_* - p)^{1/2} p_*^{-5/6} u\,(s p_*^{1/3}) \sin 2\,(\varphi + \text{const}) \tag{4.8}$$

in which the dependence of the dimensionless value of u on $t = s p_*^{1/3}$ is tabulated below ($t_0 = \tau p_*^{1/3}$)

t	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
$10^4 u$	0	50	199	448	795	1240	1779	2407	3113
t	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	$t_0 = 1.631$
$10^4 u$	3884	4694	5513	6299	7007	7585	7988	8180	8194

The form of the critical axisymmetric equilibrium figure (bifurcation figure) can be determined using elliptic integrals /2,5/ or numerically.

The two parameters ν (or p, since volume v is a given quantity) and λ each of which defines the equilibrium state (the additional parameter that determines the turn of nonaxisymmetric figures about the z axis is immaterial) were already mentioned in Sect.1. The equilibrium state can be considered to be the fixed point of functional $U_1 = \sigma\Sigma - \omega^2 I/2$ or $U_2 = \sigma\Sigma + \mu^2/(2I)$ (Σ is the area of the liquid free surface); depending on this, either p or λ appear in a natural way. The first functional is applied in investigations of relative equilibrium of nonconservative system which maintain a constant angular velocity of rotation also in perturbed motion. The second one is used in investigations of steady motions of a free system, when the moment of momentum relative to the axis of rotation is constant /8/. The considered here system is free, hence λ is to be taken as the characteristic parameter.

In the above reasoning p was taken as the parameter. This was done not only for convenience. Data obtained in terms of p together with those below enable us to carry out full comparison with conclusions of the theory of equilibrium figures of equilibrium of a rotating self-gravitating liquid, related to the properties of Jacobi's ellipsoids and their difference from the set of Maclaurin ellipsoids /9/.

Let us pass from p to λ. We represent λ in the form

$$\lambda = 2pT^2 v^{-7/3}, \qquad T = \int\limits_\Omega r^2\, d\Omega$$

where Ω is the region occupied by the liquid. The increment of λ toward its critical value λ_* on nonaxisymmetric figures is

$$\Delta\lambda = 2v^{-7/3}\{p_*[2T_*\Delta T + (\Delta T)^2] + \Delta p\,[T_*^2 + 2T_*\Delta T + (\Delta T)^2]\}$$

$$\Delta p = p - p_*, \qquad \Delta T = \int\limits_\Gamma \Big[Nr^2 - \frac{1}{2}N^2 r(3z' + r\beta') +$$

$$N^3 z'(z' + r\beta') - \frac{1}{4}N^4 z'^2\Big(\frac{z'}{r} + 3\beta'\Big) + \frac{1}{5}N^5\frac{z'^3}{r}\beta'\Big]\, d\Gamma$$

where T_* is the value of T for the bifurcation figure.

Taking into account (3.1), (4.1), (4.2), and (4.5) we obtain

$$\Delta T = \varepsilon^2 4\pi \int\limits_0^\tau \Big\{Q_1^2\,[C_1 u_0 + b_1 + D_1(C_3 u_0 + b_3) -$$

$$\frac{1}{4}\Big(\frac{3z'}{r} + \beta'\Big)u^2\Big] + C_2 u_0 + b_2 - D_2(G_3 u_0 + b_3)\Big\}\, r^3\, ds + O(\varepsilon^3)$$

As the result of calculations we obtained $T_* = 3.142 p_*^{-5/3}$, $\lambda_* = 0.412$, $\Delta T = 3.672 p_*^{-8/3}\varepsilon^2 + O(\varepsilon^3)$, hence

$$\Delta\lambda = -0.550\,\Delta p/p_* + O(\varepsilon^3) \tag{4.9}$$

The fixed points of functionals U_1 and U_2, as well as the critical axisymmetric equilibrium states /4,5/, coincide. Hence for determining the form of the branched-off figures it is sufficient to substitute in (4.8) the obtained in (4.9) quantity $1.818 p_*(\lambda - \lambda_*)$ for $p_* - p$.

We shall judge the stability of the branched-off figures by the pattern of their branching in conformity with the Poincaré–Schwarzschild reasoning /9/. A set of axisymmetric figures exists on both sides of λ_*, with $\lambda < \lambda_*$ corresponding to stable states. Branched-off figures exist when $\Delta p < 0$, hence according to (4.9) we have $\Delta\lambda > 0$. Thus branching is directed toward $\lambda > \lambda_*$ where axisymmetric states are unstable. This implies that in the neighborhood of λ_* the nonaxisymmetric figures (as well as the bifurcation figure) are stable. This means that only nonaxisymmetric figures of equilibrium may be observed when $\lambda > \lambda_*$ and ($\mu > \mu_* = 4.443 \sqrt{\rho\sigma}\, p_*^{-7/6}$).

Note that in stability investigations a correct selection of the parameter which defines equilibrium is particularly important. In the case of a nonconservative system with p as the determining parameter, from the pattern of branching would follow the conclusion of instability of branched-off figures in the neighborhood of p_*.

The results of this investigation together with previous studies of axisymmetric equilibrium forms (including ring-shaped /10/) and of their stability show a remarkable similarity of behavior of equilibrium figures of a rotating liquid with surface tension and of a rotating self-gravitating liquid /9/.

The author thanks N.S. Shcherbakov for carrying out calculations and A.D. Tiuntsev for discussions.

REFERENCES

1. WANG T.G., SAFFREN M.M. and ELLEMAN D.D., Drop dynamics in space. In: Mater. Sci. Space Appl. Space Process., New York, 1977.
2. CHANDRASEKHAR S., The stability of a rotating liquid drop. Proc. Roy. Soc., London. Ser. A, Vol.286, No.1404, 1965.
3. SLOBOZHANIN L.A., Hydrostatics in weak force fields. Equilibrium surface forms of liquid rotating in conditions of weightlessness. Izv. Akad. Nauk SSSR, MZhG, No.5, 1966.
4. BELIAEVA M.A., SLOBOZHANIN L.A. and TIUPTSOV A.D., Hydrostatics in weak force fields. In: Introduction to Dynamics of a Body with Liquid Under Conditions of Weightlessness.Moscow, VTs Akad. Nauk SSSR, 1968.
5. BABSKII V.G., KOPACHEVSKII N.D., MYSHKIS A.D., SLOBOZHANIN L.A. and TIUPTSOV A.D., The Hydromechanics of Weightlessness. Moscow, NAUKA, 1976.
6. POGORELOV A.V., Differential Geometry. Moscow, NAUKA, 1969.
7. VAINBERG M.M. and TRENOGIN V.A., Theory of Branching of Solutions of Nonlinear Equations. Moscow, NAUKA, 1969.
8. MOISEEV N.I. and RUMIANTSEV V.V., Dynamics of a Body with Cavities Containing Liquid. Moscow, NAUKA, 1965.
9. APPEL I., Equilibrium Forms of a Rotating Homogeneous Liquid. Leningrad–Moscow, ONTI,1936.
10. SLOBOZHANIN L.A., Hydrostatics in weak force fields. On ring-shaped equilibrium figures of a rotating liquid and their stability. Izv. Akad. Nauk SSSR, NZhG, No.4, 1968.

Translated by J.J.D.